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Bogoyavlensky–Toda systems of type D_N

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Abstract
In this paper we confirm a conjecture of Flaschka which relates the hierarchy of Poisson brackets and master symmetries for the Bogoyavlesky–Toda systems of type D_n with fundamental invariants of the corresponding Lie group. The areas investigated include master symmetries, recursion operators, higher Poisson brackets and invariants. The results are presented both in Flaschka coordinates (a, b) and in the natural (q, p) coordinates.

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1. Introduction

This work is the final step in a series of papers which deal with the multi-Hamiltonian structure of the Bogoyavlensky–Toda systems corresponding to the classical simple Lie groups of types A_n, B_n, C_n and D_n. The Toda lattice is a system of particles on the line where each particle interacts with its neighbour with an exponential force. The original Toda system with an infinite number of particles was considered by Toda [36] in 1967. The integrability of the system is due to Flaschka [12], Henon [16] and Manakov [25], all in 1974. The explicit solution of the finite lattice is due to Moser [27] in 1975. We restrict our attention to the finite, non-periodic version of the Toda lattice. The multiple Hamiltonian structure of the Toda lattice was established in [5–7], using master symmetries. The master symmetries were used to generate nonlinear Poisson brackets and higher order invariants. For the definition and examples of master symmetries see [14, 30, 15].

The Toda lattice corresponds to a simple Lie algebra of type A_n, i.e. $sl(n, \mathbf{C})$. We have to point out that all computations in [6, 7] are performed in $gl(n, \mathbf{C})$; it turns out that a second quadratic bracket for $sl(n, \mathbf{C})$ Toda does not exist. These results were duplicated in (q, p) coordinates by Das and Okubo [10], and Fernandes [11]. In principle, their method is general and may work for other finite-dimensional systems as well. The procedure is as follows: one defines a second Poisson bracket in the space of canonical variables $(q_1, \ldots, q_n, p_1, \ldots, p_n)$.

This gives rise to a recursion operator. The presence of a conformal symmetry as defined in Oevel [31] allows one, by using the recursion operator, to generate an infinite sequence of master symmetries. These, in turn, project onto the space of the new variables (a, b) to produce a sequence of master symmetries in the reduced space. This approach was also used in [29] in the case of the relativistic Toda lattice.

The analogous results for B_n Bogoyavlensky–Toda were computed in [7] in Flaschka coordinates, and in [17] in (q, p) coordinates. The C_n case is in [8] in Flaschka coordinates, and [17] in natural (q, p) coordinates. In this paper, we deal with the D_n case which is the most interesting and at the same time the most difficult. It is interesting because the exponents of a simple Lie group of type D_n do not form a simple arithmetical progression as in the A_n, B_n and C_n cases and it is more difficult due to the complicated structure of the root system of D_n. As in the previous cases, the exponents of the Lie group appear through the action of the master symmetries on the Hamiltonian vector fields.

The equations for the Toda systems under consideration will be written in the form

$$\dot{L}(t) = [B(t), L(t)]. \tag{1}$$

The pair of matrices L, B is known as a Lax pair. In the case of the finite nonperiodic Toda lattice L is a symmetric tridiagonal matrix and B is the projection onto the skew-symmetric part in the decomposition of L into skew-symmetric and lower triangular. In the case of Bogoyavlendky–Toda systems the matrix L will lie in the corresponding Lie algebra and B will again be obtained from L by some projection associated with a decomposition of the Lie algebra. The decomposition plays an important role in the solution of the equations by factorization.

In the case of a Toda lattice the Lax equation is obtained by the use of a transformation due to Flaschka [12] which changes the original (q, p) variables to new reduced variables (a, b). The symplectic bracket in the variables (q, p) transforms to a degenerate Poisson bracket in the variables (a, b). This linear bracket is an example of a Lie–Poisson bracket. The functions $H_n = \frac{1}{n} \operatorname{tr} L^n$ are in involution. A Lie algebraic interpretation of this bracket can be found in [19]. We denote this bracket by π_1. A quadratic Toda bracket, which we call π_2, appeared in a paper of Adler [1]. It is a Poisson bracket in which the Hamiltonian vector field generated by H_1 is the same as the Hamiltonian vector field generated by H_2 with respect to the π_1 bracket. This is an example of a bi-Hamiltonian system, an idea introduced by Magri [23]. For further details on bi-Hamiltonian systems relevant to Toda-type systems see [11, 35, 34]. A cubic bracket was found by Kupershmidt [21] via the infinite Toda lattice. We found the explicit formulae for both the quadratic and cubic brackets in some lecture notes by Flaschka. The Lenard relations are also in these notes. The Lenard relations show that the system is bi-Hamiltonian. In a situation like this, if one of the tensors is invertible one can find a recursion operator by inverting the symplectic tensor. The recursion operator is then applied to the initial symplectic bracket to produce an infinite sequence. However, in the case of a Toda lattice (in Flaschka variables (a, b)) both operators are non-invertible and therefore this method fails. The absence of a recursion operator for the finite Toda lattice is also mentioned in Morosi and Tondo [26] where a Nijenhuis tensor for the infinite Toda lattice is calculated.

The Toda lattice has been generalized in several directions. In this paper we consider the generalized Toda systems defined by Bogoyavlensky [2] and studied in Kostant [19] and Olshanetsky and Perelomov [33]. The Toda system is generalized to the tridiagonal coadjoint orbit of the Borel subgroup of an arbitrary simple Lie group. Therefore, for each simple Lie group there is a corresponding mechanical system of Toda type. We will call such systems the Bogoyavlensky–Toda systems. We have to point out that the original paper [2] deals with the periodic version of the system. The methods of this paper can be used with only minor modifications to obtain similar results for the periodic Toda lattices but in that case the Lie algebraic setting is Kac–Moody algebras instead of simple Lie algebras.

There is also an interesting connection with the corresponding generalized Volterra systems also defined by Bogoyavlensky [3]. It seems that the multiple Hamiltonian structures of the Volterra and Toda lattices are in one-to-one correspondence. Multiple Hamiltonian structures for the generalized Volterra lattices were constructed recently by Kouzaris [20] at least for the classical Lie algebras. The relation between the Volterra systems of types B_n and C_n and the corresponding Toda B_n, C_n systems is demonstrated in the forthcoming paper [9]. The connection between Volterra D_n and Toda D_n is still an open problem.

In section 2 we present the necessary background on bi-Hamiltonian systems and master symmetries. We also define the exponents of a simple Lie group.

Section 3 is a review of the classical finite nonperiodic Toda lattice. This system was investigated in [36, 12, 13, 16, 25, 18, 28, 27]. We define the quadratic and higher Toda brackets and show that they satisfy certain Lenard-type relations. We briefly describe the construction of master symmetries and the new Poisson brackets as in [6, 7]. We also describe the method of Fernandes [11].

In section 4, we define the integrable Bogoyavlensky–Toda systems associated with simple Lie groups. We present in detail the system of type D_n. We will not present the results for the B_n, C_n systems but we refer the reader to [7, 8, 17]. We make the computations both in Flaschka coordinates (a, b) and also in (q, p) variables. In each case we compute invariants, Poisson tensors, recursion operators and master symmetries. There is a sequence of invariants $H_2, H_4, \ldots$ of even degree and an additional invariant of degree n. Let χ_i denote the Hamiltonian vector field generated by H_i and let Z_0 denote a conformal symmetry. Then we have

$$[Z_0, \chi_j] = f(j)\chi_j.$$

The values of $f(j)$ corresponding to independent χ_j generate all the exponents except one. When Z_0 acts on the Hamiltonian vector field χ_P, where P is the invariant corresponding to the Pfaffian of the Jacobi matrix, we obtain the last exponent $n-1$. For example, in the case of D_5 the exponents are 1, 3, 5, 7 and 4. The independent invariants are H_2, H_4, H_6, H_8 and P_5 where $H_{2i} = \frac{1}{2i}\operatorname{Tr} L^{2i}$ and $P_5 = \sqrt{\det L}$. We obtain

$$[Z_0, \chi_2] = \chi_2 \qquad [Z_0, \chi_4] = 3\chi_4 \qquad [Z_0, \chi_6] = 5\chi_6$$
$$[Z_0, \chi_8] = 7\chi_8 \qquad [Z_0, \chi_{P_5}] = 4\chi_{P_5}.$$

In other words, the coefficients on the right-hand side are precisely the exponents of a simple Lie group of type D_5.

2. Background

We assume that the reader is familiar with the concept of Poisson manifold. See for example [22, 37, 38]. Let M be a C^∞ manifold equipped with two Poisson tensors π_0 and π_1. The two tensors are called compatible if $\pi_0 + \pi_1$ is Poisson. If π_0 is symplectic, we call the Poisson pair (π_0, π_1) non-degenerate. In this case, the $(1, 1)$-tensor $\mathcal{R}$ defined by

$$\mathcal{R} = \pi_1\pi_0^{-1} \tag{2}$$

is called the *recursion operator* associated with the non-degenerate pair. Recursion operators were introduced by Olver [32].

A *bi-Hamiltonian system* is defined by specifying two Hamiltonian functions H_0, H_1 satisfying

$$\pi_0\nabla H_1 = \pi_1\nabla H_0 \tag{3}$$

where π_i, $i = 0, 1$, denotes the Poisson matrix of the tensor π_i. The theory of bi-Hamiltonian systems was developed by Magri [23]. He established the existence of a hierarchy of mutually commuting functions $H_0, H_1, \ldots,$ all in involution with respect to both brackets. They generate mutually commuting bi-Hamiltonian flows χ_i satisfying the Lenard recursion relations. For more details see [24].

We recall the definition and basic properties of master symmetries. Consider a differential equation on a manifold M, defined by a vector field χ. We are mostly interested in the case where χ is a Hamiltonian vector field. A vector field Z is a *symmetry* of the equation if

$$[Z, \chi] = 0.$$

A vector field Z will be called a *master symmetry* if

$$[[Z, \chi], \chi] = 0$$

but

$$[Z, \chi] \neq 0.$$

Suppose that we have a bi-Hamiltonian system defined by the Poisson tensors π_0, π_1 and the Hamiltonians H_0, H_1. Assume that π_0 is symplectic and let $\chi_1 = \chi$. We define the recursion operator $\mathcal{R} = \pi_1\pi_0^{-1}$, the higher flows

$$\chi_i = \mathcal{R}^{i-1}\chi_1$$

and the higher order Poisson tensors

$$\pi_i = \mathcal{R}^i\pi_0.$$

For a non-degenerate bi-Hamiltonian system, master symmetries can be generated using a method due to Oevel [31].

Theorem 1 (Oevel)**.** *Suppose that Z_0 is a conformal symmetry for both π_0, π_1 and H_0, i.e., for some scalars λ, μ and ν we have*

$$\mathcal{L}_{Z_0}\pi_0 = \lambda\pi_0 \qquad \mathcal{L}_{Z_0}\pi_1 = \mu\pi_1 \qquad \mathcal{L}_{Z_0}H_0 = \nu H_0. \tag{4}$$

Then the vector fields

$$Z_i = \mathcal{R}^i Z_0 \tag{5}$$

are master symmetries and we have

$$(a) \quad [Z_i, \chi_j] = (\mu + \nu + (j-1)(\mu - \lambda))\chi_{i+j} \tag{6}$$

$$(b) \quad [Z_i, Z_j] = (\mu - \lambda)(j - i)Z_{i+j} \tag{7}$$

$$(c) \quad \mathcal{L}_{Z_i}\pi_j = (\mu + (j - i - 1)(\mu - \lambda))\pi_{i+j} \tag{8}$$

$$(d) \quad \mathcal{L}_{Z_i}H_j = (\nu + (j + i)(\mu - \lambda))H_{i+j}. \tag{9}$$

Finally, let us recall the definition of exponents for a semi-simple group G. An excellent reference is the book by Collingwood and McGovern [4]. Let G be a connected complex simple Lie group G. We form the de Rham cohomology groups $H^i(G, \mathbf{C})$ and the corresponding Poincaré polynomial of G:

$$p_G(t) = \sum d_i t^i$$

where $d_i = \dim H^i(G, \mathbf{C})$. A theorem of Hopf shows that the cohomology algebra is a finite product of l spheres of odd dimension where l is the rank of G. This theorem implies that

$$p_G(t) = \prod_{i=1}^{l}(1 + t^{2e_i+1}).$$

The positive integers $\{e_1, e_2, \ldots, e_l\}$ are called the *exponents* of G. One can also extract the exponents from the root space decomposition of G. The connection with the invariant polynomials is as follows: let $H_1, H_2, \ldots, H_l$ be independent, homogeneous, invariant polynomials of degrees $m_1, m_2, \ldots, m_l$. Then $e_i = m_i - 1$. The exponents of a simple Lie group are given in the following list:

A_{n-1}:	$1, 2, 3, \ldots, n-1$
B_n, C_n:	$1, 3, 5, \ldots, 2n-1$
D_n:	$1, 3, 5, \ldots, 2n-3, n-1$
G_2:	$1, 5$
F_4:	$1, 5, 7, 11$
E_6:	$1, 4, 5, 7, 8, 11$
E_7:	$1, 5, 7, 9, 11, 13, 17$
E_8:	$1, 7, 11, 13, 17, 19, 23, 29.$

3. Finite, non-periodic A_n Toda lattice

In this section we review the classical Toda lattice before moving to the D_n case in section 4. The Toda lattice is a completely integrable classical mechanical system consisting of n particles on the line and subject to a system of springs which behave exponentially. The Hamiltonian function of the system is

$$H(q_1, \ldots, q_n, p_1, \ldots, p_n) = \sum_{j=1}^{n} \frac{1}{2} p_j^2 + \sum_{j=1}^{n-1} \mathrm{e}^{q_j - q_{j+1}} \tag{10}$$

where $q_j(t)$ is the position of the jth particle and $p_j(t)$ is the corresponding momentum. Hamilton's equations are

$$\dot{q}_j = p_j \qquad \dot{p}_j = \mathrm{e}^{q_{j-1} - q_j} - \mathrm{e}^{q_j - q_{j+1}}. \tag{11}$$

To determine the set of independent functions $\{H_1, \ldots, H_n\}$ which are constants of motion for Hamilton's equations, one uses Flaschka's transformation:

$$a_i = \tfrac{1}{2} \mathrm{e}^{\frac{1}{2}(q_i - q_{i+1})} \qquad b_i = -\tfrac{1}{2} p_i. \tag{12}$$

Then,

$$\dot{a}_i = a_i(b_{i+1} - b_i) \qquad \dot{b}_i = 2\big(a_i^2 - a_{i-1}^2\big). \tag{13}$$

These equations can be written as a Lax pair $\dot{L} = [B, L]$, where L is the symmetric Jacobi matrix

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & \cdots & 0 \\ a_1 & b_2 & a_2 & \cdots & & \vdots \\ 0 & a_2 & b_3 & \ddots & & \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & a_{n-1} \\ 0 & \cdots & & \cdots & a_{n-1} & b_n \end{pmatrix} \tag{14}$$

and

$$B = \begin{pmatrix} 0 & a_1 & 0 & \cdots & \cdots & 0 \\ -a_1 & 0 & a_2 & \cdots & & \vdots \\ 0 & -a_2 & 0 & \ddots & & \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & a_{n-1} \\ 0 & \cdots & \cdots & & -a_{n-1} & 0 \end{pmatrix}. \tag{15}$$

It follows that the functions $H_i = \frac{1}{i} \operatorname{tr} L^i$ are constants of motion.

Consider $\mathbf{R}^{2n}$ with coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$, the standard symplectic bracket and the Flaschka transformation $F : \mathbf{R}^{2n} \to \mathbf{R}^{2n-1}$ defined by

$$F : (q_1, \ldots, q_n, p_1, \ldots, p_n) \to (a_1, \ldots, a_{n-1}, b_1, \ldots, b_n). \tag{16}$$

The standard symplectic bracket on $\mathbf{R}^{2n}$ reduces, under the mapping F, to a linear bracket on $\mathbf{R}^{2n-1}$ determined by

$$\{a_i, b_i\} = -a_i \qquad \{a_i, b_{i+1}\} = a_i \tag{17}$$

all other brackets are zero. We denote this Poisson tensor by π_1. The only Casimir is $H_1 = b_1 + b_2 + \cdots + b_n$. The Hamiltonian turns out to be $H_2 = \frac{1}{2} \operatorname{tr} L^2$ and the functions H_j are in involution.

There is also a quadratic bracket π_2 which appeared in a paper of Adler [1] in 1979. The defining relations for the new bracket π_2 are:

$$\begin{aligned} &\{a_i, a_{i+1}\} = \tfrac{1}{2} a_i a_{i+1} && \{a_i, b_i\} = -a_i b_i \\ &\{a_i, b_{i+1}\} = a_i b_{i+1} && \{b_i, b_{i+1}\} = 2a_i^2 \end{aligned} \tag{18}$$

all other brackets are zero. This bracket has $\det L$ as Casimir and $H_1 = \operatorname{tr} L$ is the Hamiltonian. The eigenvalues of L are still in involution and

$$\pi_2 \nabla \lambda_j = \lambda_j \pi_1 \nabla \lambda_j \qquad \forall j.$$

It easily follows that

$$\pi_2 \nabla H_j = \pi_1 \nabla H_{j+1} \qquad \forall j. \tag{19}$$

These relations are similar to the Lenard relations for the KdV equation. They show that the Toda lattice is a bi-Hamiltonian system.

Since it was impossible to find a recursion operator for the non-periodic Toda lattice (at least in Flaschka coordinates) a different method was used to generate invariants. The idea was to define master symmetries, and use Lie derivatives to generate higher invariants.

We describe the construction following [6, 7]. We denote the master symmetries by X_n. These vector fields generate an infinite sequence of contravariant 2-tensors π_n, for $n \geqslant 1$. We summarize the properties of X_n and π_n.

Theorem 2.

(i) π_n *are all Poisson.*
(ii) The functions $H_n = \frac{1}{n} \operatorname{tr} L^n$ *are in involution with respect to all of the* π_n.
(iii) $X_n(H_m) = (n + m) H_{n+m}$.
(iv) $\mathcal{L}_{X_n} \pi_m = (m - n - 2) \pi_{n+m}$.
(v) $\pi_n \nabla H_l = \pi_{n-1} \nabla H_{l+1}$, *where* π_n *denotes the Poisson matrix of the tensor* π_n.

To define the vector fields X_n we consider expressions of the form

$$\dot{L} = [B, L] + L^n. \tag{20}$$

This equation is similar to a Lax equation, but in this case the eigenvalues satisfy $\dot{\lambda} = \lambda^n$ instead of $\dot{\lambda} = 0$.

There is another method of finding the master symmetries due to Fernandes [11] which we describe briefly. The first step is to define a second Poisson bracket on the space of canonical variables $(q_1, \ldots, q_n, p_1, \ldots, p_n)$. This bracket appears in Das and Okubo [10] and Fernandes [11]. We follow the notation from [11]. Let J_0 be the symplectic bracket on $\mathbf{R}^{2n}$ and define J_1 as follows:

$$\{q_i, q_j\} = 1 \qquad \{p_i, q_i\} = p_i \qquad \{p_i, p_{i+1}\} = \mathrm{e}^{q_i - q_{i+1}} \tag{21}$$

all other brackets are zero. Also define

$$h_0 = \sum_{i=1}^{n} p_i \qquad h_1 = \sum_{i=1}^{n} \frac{p_i^2}{2} + \sum_{i=1}^{n-1} \mathrm{e}^{q_i - q_{i+1}}. \tag{22}$$

Since we have a non-degenerate pair (J_0, J_1), there exists a recursion operator defined by $\mathcal{R} = J_1 J_0^{-1}$. It follows easily that the vector field

$$Z_0 = \sum_{i=1}^{n} \frac{n+1-2i}{2} \frac{\partial}{\partial q_i} + \sum_{i=1}^{n} p_i \frac{\partial}{\partial p_i} \tag{23}$$

is a conformal symmetry for J_0, J_1 and h_0 and therefore, Oevel's theorem applies. The constants in theorem 1 turn out to be $\lambda = -1$, $\mu = 0$ and $\nu = 1$. We end up with the following relations:

$$[Z_i, \chi_j] = j\chi_{i+j} \tag{24}$$

$$\mathcal{L}_{Z_i} J_j = (j - i - 1) J_{i+j} \tag{25}$$

$$[Z_i, Z_j] = (j - i) Z_{i+j}. \tag{26}$$

Taking into account the way we defined the linear bracket π_1 on $\mathbf{R}^{2n-1}$, the mapping F is a Poisson mapping between J_0 and π_1. But it is also a Poisson mapping between J_1 and π_2. In fact, the Poisson tensor J_1 reduces, under the mapping F, to π_2. The Hamiltonians h_0 and h_1 correspond to the reduced Hamiltonians H_1 and H_2 respectively. The recursion operator $\mathcal{R}$ cannot be reduced. Actually, it is easy to see that there exists no recursion operator in the reduced space. The kernels of the two Poisson structures π_1 and π_2 are different and, therefore, it is impossible to find an operator that maps one to the other.

The deformation relations (24)–(26) also reduce and we obtain the deformation relations of theorem 2. Note that (24) gives a procedure for generating the exponents of a simple Lie group of type A_n.

4. Bogoyavlensky–Toda systems of type D_N

4.1. *Definition of the systems*

In this section we consider mechanical systems which generalize the finite, nonperiodic Toda lattice. These systems correspond to Dynkin diagrams. It is well known that irreducible root systems classify simple Lie groups. So, in this generalization for each simple Lie group there exists a mechanical system of Toda type.

The generalization is obtained from the following simple observation: in terms of the natural basis q_i of weights, the simple roots of A_{n-1} are

$$q_1 - q_2, q_2 - q_3, \ldots, q_{n-1} - q_n.$$

On the other hand, the potential for the Toda lattice is of the form

$$\mathrm{e}^{q_1-q_2} + \mathrm{e}^{q_2-q_3} + \cdots + \mathrm{e}^{q_{n-1}-q_n}.$$

We note that the angle between $q_{i-1} - q_i$ and $q_i - q_{i+1}$ is $\frac{2\pi}{3}$ and the lengths of $q_i - q_{i+1}$ are all equal. The Toda lattice corresponds to a Dynkin diagram of type A_{n-1}.

More generally, we consider potentials of the form

$$U = c_1\,\mathrm{e}^{f_1(q)} + \cdots + c_l\,\mathrm{e}^{f_l(q)}$$

where $c_1, \ldots, c_l$ are constants, $f_i(q)$ is linear and l is the rank of the simple Lie group. For each Dynkin diagram we construct a Hamiltonian system of Bogoyavlensky–Toda type. These systems are interesting not only because they are integrable, but also for their fundamental importance in the theory of semisimple Lie groups. For example, Kostant in [19] shows that the integration of these systems and the theory of the finite-dimensional representations of semisimple Lie groups are equivalent.

For reference, we give a complete list of the Hamiltonians for each simple Lie algebra.

$$A_{n-1}: \quad H = \frac{1}{2}\sum_1^n p_j^2 + \mathrm{e}^{q_1-q_2} + \cdots + \mathrm{e}^{q_{n-1}-q_n}$$

$$B_n: \quad H = \frac{1}{2}\sum_1^n p_j^2 + \mathrm{e}^{q_1-q_2} + \cdots + \mathrm{e}^{q_{n-1}-q_n} + \mathrm{e}^{q_n}$$

$$C_n: \quad H = \frac{1}{2}\sum_1^n p_j^2 + \mathrm{e}^{q_1-q_2} + \cdots + \mathrm{e}^{q_{n-1}-q_n} + \mathrm{e}^{2q_n}$$

$$D_n: \quad H = \frac{1}{2}\sum_1^n p_j^2 + \mathrm{e}^{q_1-q_2} + \cdots + \mathrm{e}^{q_{n-1}-q_n} + \mathrm{e}^{q_{n-1}+q_n}$$

$$G_2: \quad H = \frac{1}{2}\sum_1^3 p_j^2 + \mathrm{e}^{q_1-q_2} + \mathrm{e}^{-2q_1+q_2+q_3}$$

$$F_4: \quad H = \frac{1}{2}\sum_1^4 p_j^2 + \mathrm{e}^{q_1-q_2} + \mathrm{e}^{q_2-q_3} + \mathrm{e}^{q_3} + \mathrm{e}^{\frac{1}{2}(q_4-q_1-q_2-q_3)}$$

$$E_6: \quad H = \frac{1}{2}\sum_1^8 p_j^2 + \sum_1^4 \mathrm{e}^{q_j-q_{j+1}} + \mathrm{e}^{-(q_1+q_2)} + \mathrm{e}^{\frac{1}{2}(-q_1+q_2+\cdots+q_7-q_8)}$$

$$E_7: \quad H = \frac{1}{2}\sum_1^8 p_j^2 + \sum_1^5 \mathrm{e}^{q_j-q_{j+1}} + \mathrm{e}^{-(q_1+q_2)} + \mathrm{e}^{\frac{1}{2}(-q_1+q_2+\cdots+q_7-q_8)}$$

$$E_8: \quad H = \frac{1}{2}\sum_1^8 p_j^2 + \sum_1^6 \mathrm{e}^{q_j-q_{j+1}} + \mathrm{e}^{-(q_1+q_2)} + \mathrm{e}^{\frac{1}{2}(-q_1+q_2+\cdots+q_7-q_8)}.$$

4.2. A recursion operator for D_n Bogoyavlensky–Toda systems in Flaschka coordinates

In this subsection, we show that higher polynomial brackets exist also in the case of D_n Bogoyavlensky–Toda systems. Using Flaschka coordinates, we will prove that these systems possess a recursion operator and we will construct an infinite sequence of compatible Poisson brackets in which the constants of motion are in involution.

The Hamiltonian for D_n is

$$H = \frac{1}{2}\sum_{1}^{n} p_j^2 + \mathrm{e}^{q_1-q_2} + \cdots + \mathrm{e}^{q_{n-1}-q_n} + \mathrm{e}^{q_{n-1}+q_n} \qquad n \geqslant 4. \tag{27}$$

We make a Flaschka-type transformation, $F : \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ defined by

$$F : (q_1, \ldots, q_n, p_1, \ldots, p_n) \to (a_1, \ldots, a_n, b_1, \ldots, b_n)$$

with

$$\begin{aligned} a_i &= \tfrac{1}{2}\,\mathrm{e}^{\frac{1}{2}(q_i-q_{i+1})} && i = 1, 2, \ldots, n-1 \quad a_n = \tfrac{1}{2}\,\mathrm{e}^{\frac{1}{2}(q_{n-1}+q_n)} \\ b_i &= -\tfrac{1}{2}p_i && i = 1, 2, \ldots, n. \end{aligned} \tag{28}$$

Then

$$\begin{aligned} \dot{a}_i &= a_i(b_{i+1} - b_i) && i = 1, 2, \ldots, n-1 \\ \dot{a}_n &= -a_n(b_{n-1} + b_n) \\ \dot{b}_i &= 2\big(a_i^2 - a_{i-1}^2\big) && i = 1, 2, \ldots, n-2 \quad \text{and} \quad i = n \\ \dot{b}_{n-1} &= 2\big(a_n^2 + a_{n-1}^2 - a_{n-2}^2\big). \end{aligned} \tag{29}$$

These equations can be written as a Lax pair $\dot{L} = [B, L]$, where L is the symmetric matrix

$$\begin{pmatrix} b_1 & a_1 & & & & & & & \\ a_1 & \ddots & \ddots & & & & & & \\ & \ddots & \ddots & a_{n-1} & -a_n & 0 & & & \\ & & a_{n-1} & b_n & 0 & a_n & & & \\ & & -a_n & 0 & -b_n & -a_{n-1} & & & \\ & & 0 & a_n & -a_{n-1} & \ddots & \ddots & & \\ & & & & & \ddots & \ddots & -a_1 \\ & & & & & & -a_1 & -b_1 \end{pmatrix} \tag{30}$$

and B is the skew-symmetric part of L (in the decomposition, lower Borel plus skew-symmetric).

The mapping $F : \mathbf{R}^{2n} \to \mathbf{R}^{2n}, (q_i, p_i) \to (a_i, b_i)$, defined by (28), transforms the standard symplectic bracket J_0 into another symplectic bracket π_1 given by

$$\begin{aligned} \{a_i, b_i\} &= -a_i && i = 1, 2, \ldots, n \\ \{a_i, b_{i+1}\} &= a_i && i = 1, 2, \ldots, n-1 \\ \{a_n, b_{n-1}\} &= -a_n. \end{aligned} \tag{31}$$

We obtain a hierarchy of invariant polynomials, which we denote by

$$H_2, H_4, \ldots, H_{2n}, \ldots$$

defined by $H_{2i} = \frac{1}{2i}\operatorname{Tr} L^{2i}$. The degrees of the first $n-1$ (independent) polynomials are $2, 4, \ldots, 2n-2$ and the exponents of the corresponding Lie group are $1, 3, \ldots, 2n-3$. We also define

$$P_n = \sqrt{\det L}.$$

Since L is a $2n \times 2n$ matrix, the degree of $\det L$ is $2n$ and therefore the degree of P_n is n. The corresponding exponent is $n-1$.

We look for a bracket π_3 which satisfies

$$\pi_3 \nabla H_2 = \pi_1 \nabla H_4. \tag{32}$$

We define the following homogeneous cubic bracket π_3 whose non-zero terms are:

$$\begin{aligned}
&\{a_i, a_{i+1}\} = a_i a_{i+1} b_{i+1} \qquad i = 1, 2, \ldots, n-2 \\
&\{a_{n-2}, a_n\} = a_{n-2} a_n b_{n-1} \\
&\{a_{n-1}, a_n\} = 2a_{n-1} a_n b_n \\
&\{a_i, b_i\} = -a_i\left(b_i^2 + a_i^2\right) \qquad i = 1, 2, \ldots, n-2 \\
&\{a_{n-1}, b_{n-1}\} = -a_{n-1}\left(a_{n-1}^2 + 3a_n^2 + b_{n-1}^2\right) \\
&\{a_n, b_n\} = -a_n\left(a_n^2 + b_n^2 - a_{n-1}^2\right) \\
&\{a_i, b_{i+1}\} = a_i\left(a_i^2 + b_{i+1}^2\right) \qquad i = 1, 2, \ldots, n-2 \\
&\{a_{n-1}, b_n\} = a_{n-1}\left(a_{n-1}^2 + b_n^2 - a_n^2\right) \\
&\{a_i, b_{i+2}\} = a_i a_{i+1}^2 \qquad i = 1, 2, \ldots, n-3 \\
&\{a_{n-2}, b_n\} = a_{n-2}\left(a_{n-1}^2 - a_n^2\right) \\
&\{a_i, b_{i-1}\} = -a_{i-1}^2 a_i \qquad i = 2, 3, \ldots, n-1 \\
&\{a_n, b_{n-2}\} = -a_{n-2}^2 a_n \\
&\{a_n, b_{n-1}\} = -a_n\left(3a_{n-1}^2 + a_n^2 + b_{n-1}^2\right) \\
&\{b_i, b_{i+1}\} = 2a_i^2(b_i + b_{i+1}) \qquad i = 1, 2, \ldots, n-2 \\
&\{b_{n-1}, b_n\} = 2a_{n-1}^2(b_{n-1} + b_n) + 2a_n^2(b_n - b_{n-1}).
\end{aligned} \tag{33}$$

We summarize the properties of this new bracket in the following:

Theorem 3. *The bracket π_3 satisfies:*

(1) π_3 is Poisson.

(2) π_3 is compatible with π_1.
Define $\mathcal{R} = \pi_3 \pi_1^{-1}$. Then $\mathcal{R}$ is a recursion operator. We obtain a hierarchy

$$\pi_1, \pi_3, \pi_5, \ldots$$

consisting of compatible Poisson brackets of odd degree in which the constants of motion are in involution.

(3) All the H_{2i} and P_n are in involution with respect to all the brackets $\pi_1, \pi_3, \pi_5, \ldots$.

(4) $\pi_{j+2} \nabla H_{2i} = \pi_j \nabla H_{2i+2} \quad \forall i, j.$

The proof of (1) is a straightforward verification of the Jacobi identity. We will see later, in the next subsection, that π_3 is the Lie derivative of π_1 in the direction of a master symmetry and this fact makes π_1, π_3 compatible. (4) follows from properties of the recursion operator. (3) is a consequence of (4) (see for example [7], p 5524 for a method of proof). The only part which is not obvious is the involution of P_n with H_n which will be proved at the end of the next subsection using master symmetries.

4.3. Master symmetries

We would like to make some observations concerning master symmetries. Due to the presence of a recursion operator, we will use the approach of Oevel. We define Z_0 to be the Euler vector field

$$Z_0 = \sum_{i=1}^{n} a_i \frac{\partial}{\partial a_i} + b_i \frac{\partial}{\partial b_i}.$$

We define the master symmetries Z_i by

$$Z_i = \mathcal{R}^i Z_0.$$

For obvious reasons we use the notation

$$X_{2i} = Z_i \qquad h_i = H_{2i+2} \qquad \Pi_i = \pi_{2i+1} \qquad \psi_i = \chi_{2i+2} \qquad i = 0, 1, 2, \ldots$$

where χ_{2i} denotes the Hamiltonian vector field generated by H_{2i}, with respect to π_1. This notation is convenient since X_2 is a master symmetry which raises the degrees of invariants and Poisson tensors by 2 each time. One calculates easily that

$$\mathcal{L}_{Z_0}\Pi_0 = -\Pi_0 \qquad \mathcal{L}_{Z_0}\Pi_1 = \Pi_1 \qquad \mathcal{L}_{Z_0}h_0 = 2h_0.$$

Therefore Z_0 is a conformal symmetry for Π_0, Π_1 and h_0. The constants appearing in Oevel's theorem are $\lambda = -1$, $\mu = 1$ and $\nu = 2$. Therefore we obtain

$$[Z_i, \psi_j] = (1+2j)\,\psi_{i+j} \quad \Longleftrightarrow \quad [X_{2i}, \chi_{2j+2}] = (1+2j)\chi_{2(i+j+1)} \tag{34}$$

$$[Z_i, Z_j] = 2(j-i)Z_{i+j} \quad \Longleftrightarrow \quad [X_{2i}, X_{2j}] = 2(j-i)X_{2(i+j)} \tag{35}$$

$$\mathcal{L}_{Z_i}(\Pi_j) = (2j-2i-1)\Pi_{i+j} \quad \Longleftrightarrow \quad \mathcal{L}_{X_{2i}}(\pi_{2j+1}) = (2j-2i-1)\pi_{2(i+j)+1} \tag{36}$$

$$Z_i(h_j) = (2+2i+2j)h_{i+j} \quad \Longleftrightarrow \quad X_{2i}(H_{2j}) = 2(i+j)H_{2(i+j)}. \tag{37}$$

Remark 1. The relation (36) implies that $\mathcal{L}_{X_2}(\pi_1) = -3\pi_3$ and therefore π_3 is the Lie derivative of π_1 in the direction of a master symmetry. This makes π_1 compatible with π_3 (see [7], p 5518).

Remark 2. The relation (34) gives a procedure for generating almost all the exponents. As we mentioned in the introduction, the last exponent is generated by the application of the conformal symmetry on the Hamiltonian vector field corresponding to the Pfaffian of the Jacobi matrix.

It is interesting to note that one can obtain the master symmetry X_2 by using the matrix equation

$$\dot{L} = [B, L] + L^3 \tag{38}$$

where L is the Lax matrix (30) and B is the skew-symmetric matrix defined as follows:

$$B = \left(\begin{array}{cccccccccccc}
0 & x_1 & y_1 & 0 & & & & & & & & \\
-x_1 & 0 & x_2 & y_2 & \ddots & & & & & & & \\
-y_1 & -x_2 & 0 & \ddots & \ddots & 0 & & & & 0 & & \\
0 & -y_2 & \ddots & \ddots & x_{n-2} & y_{n-2} & w & & & & & \\
 & \ddots & \ddots & -x_{n-2} & 0 & x_{n-1} & y_{n-1} & 0 & & & & \\
 & & 0 & -y_{n-2} & -x_{n-1} & 0 & 0 & -y_{n-1} & -w & & & \\
 & & & -w & -y_{n-1} & 0 & 0 & -x_{n-1} & -y_{n-2} & 0 & & \\
 & & & & 0 & y_{n-1} & x_{n-1} & 0 & -x_{n-2} & \ddots & \ddots & \\
 & & & & & w & y_{n-2} & x_{n-2} & \ddots & \ddots & -y_2 & 0 \\
 & & 0 & & & & 0 & \ddots & \ddots & 0 & -x_2 & -y_1 \\
 & & & & & & & \ddots & y_2 & x_2 & 0 & -x_1 \\
 & & & & & & & & 0 & y_1 & x_1 & 0
\end{array}\right)$$

where

$$x_i = a_i \left\{ \sum_{j=1}^{i-1} b_j + (i+1-n)\,(b_i + b_{i+1}) \right\} \qquad i = 1, 2, \ldots, n-1$$

$$y_i = (i+1-n)\, a_i a_{i+1} \qquad i = 1, 2, \ldots, n-2$$

$$y_{n-1} = -a_n \sum_{j=1}^{n-2} b_j$$

$$w = a_{n-2} a_n.$$

The matrix B was chosen in such a way that both sides of (38) have the same form. The components of the vector field X_2 are defined by the right-hand side of (38).

Finally we note the action of the first master symmetry on $P_n = \sqrt{\det L}$:

$$X_2(P_n) = P_n H_2.$$

Remark. This last result should be expected since the eigenvalues of L satisfy $\dot{\lambda} = \lambda^3$ under (38). Therefore,

$$\begin{aligned}
X_2(P_n) &= X_2(\sqrt{\det L}) \\
&= X_2(\sqrt{\lambda_1 \cdots \lambda_n}) \\
&= \frac{1}{2}(\lambda_1 \cdots \lambda_n)^{-\frac{1}{2}} (\dot{\lambda}_1 \lambda_2 \cdots \lambda_n + \lambda_1 \dot{\lambda}_2 \cdots \lambda_n + \cdots + \lambda_1 \cdots \dot{\lambda}_n) \\
&= \frac{1}{2\sqrt{\det L}} \left(\lambda_1^3 \lambda_2 \cdots \lambda_n + \lambda_1 \lambda_2^3 \cdots \lambda_n + \cdots + \lambda_1 \cdots \lambda_n^3\right) \\
&= \frac{\det L}{2\sqrt{\det L}} \left(\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2\right) \\
&= \sqrt{\det L}\, \frac{\left(\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2\right)}{2} \\
&= P_n H_2.
\end{aligned}$$

We conclude this subsection by proving the involution of H_i with P_n. It is clearly enough to show the involution of the eigenvalues of L since P_n and H_i are both functions of the eigenvalues. It is well known that the eigenvalues are in involution with respect to the symplectic bracket π_1. We will give here a proof based on the Lenard relations (32). Let λ and μ be two distinct eigenvalues and let U, V be the gradients of λ and μ respectively. We use the notation $\{\, ,\, \}$ to denote the bracket π_1 and $\langle\, ,\, \rangle$ the standard inner product. The Lenard relations (32) translate into $\pi_3 U = \lambda^2 \pi_1 U$ and $\pi_3 V = \mu^2 \pi_1 V$. Therefore,

$$\begin{aligned}
\{\lambda, \mu\} &= \langle \pi_1 U, V \rangle \\
&= \frac{1}{\lambda^2} \langle \pi_3 U, V \rangle \\
&= -\frac{1}{\lambda^2} \langle U, \pi_3 V \rangle \\
&= -\frac{1}{\lambda^2} \langle U, \mu^2 \pi_1 V \rangle \\
&= -\frac{\mu^2}{\lambda^2} \langle U, \pi_1 V \rangle
\end{aligned}$$

$$= \frac{\mu^2}{\lambda^2}\langle \pi_1 U, V\rangle$$
$$= \frac{\mu^2}{\lambda^2}\{\lambda, \mu\}.$$

Therefore, $\{\lambda, \mu\} = 0$. To show the involution with respect to all brackets π_{2j+1} and in view of (36) it is enough to show the following: let f_1, f_2 be two functions in involution with respect to the Poisson bracket π, let X be a vector field such that $X(f_i) = f_i^3$ for $i = 1, 2$. Define a Poisson bracket w by $w = \mathcal{L}_X\pi$. Then the functions f_1, f_2 remain in involution with respect to the bracket w. The proof follows trivially if we write $w = \mathcal{L}_X\pi$ in Poisson form:

$$\{f_1, f_2\}_w = X\{f_1, f_2\}_\pi - \{f_1, X(f_2)\}_\pi - \{X(f_1), f_2\}_\pi.$$

Remark. We have to point out that unlike the case of B_n and C_n the cubic bracket (33) was discovered not by manipulating the left-hand side of (32) but through the use of the master symmetry X_2. In other words, one constructs the master symmetry X_2 using (38) and then computes $\pi_3 = -\frac{1}{3}\mathcal{L}_{X_2}\pi_1$.

4.4. A recursion operator for D_n Toda systems in natural (q, p) coordinates

We now define a bi-Hamiltonian formulation for D_n Bogoyavlensky–Toda systems in natural (q_i, p_i) coordinates. This bracket is simply the pull-back of π_3 under the Flaschka transformation (28). After some tedious calculation, we obtain the following bracket in (q_i, p_i) coordinates:

$$\begin{aligned}
&\{q_i, q_j\} = -2p_j \qquad i < j\\
&\{q_i, p_i\} = p_i^2 + 2\,\mathrm{e}^{q_i - q_{i+1}} \qquad i = 1, 2, \ldots, n-2\\
&\{q_{n-1}, p_{n-1}\} = p_{n-1}^2 + 2\,\mathrm{e}^{q_{n-1}-q_n} + 2\,\mathrm{e}^{q_{n-1}+q_n}\\
&\{q_n, p_n\} = p_n^2\\
&\{q_i, p_{i-1}\} = \mathrm{e}^{q_{i-1}-q_i} \qquad i = 2, 3, \ldots, n-1\\
&\{q_n, p_{n-1}\} = \mathrm{e}^{q_{n-1}-q_n} - \mathrm{e}^{q_{n-1}+q_n}\\
&\{q_i, p_{i+1}\} = -\mathrm{e}^{q_i-q_{i+1}} + 2\,\mathrm{e}^{q_{i+1}-q_{i+2}} \qquad i = 1, 2, \ldots, n-3\\
&\{q_{n-2}, p_{n-1}\} = -\mathrm{e}^{q_{n-2}-q_{n-1}} + 2\,\mathrm{e}^{q_{n-1}-q_n} + 2\,\mathrm{e}^{q_{n-1}+q_n}\\
&\{q_{n-1}, p_n\} = -\mathrm{e}^{q_{n-1}-q_n} + \mathrm{e}^{q_{n-1}+q_n}\\
&\{q_i, p_j\} = -2\,\mathrm{e}^{q_{j-1}-q_j} + 2\,\mathrm{e}^{q_j-q_{j+1}} \qquad 1 \leqslant i < j-1 \leqslant n-3\\
&\{q_i, p_{n-1}\} = -2\,\mathrm{e}^{q_{n-2}-q_{n-1}} + 2\,\mathrm{e}^{q_{n-1}-q_n} + 2\,\mathrm{e}^{q_{n-1}+q_n} \qquad i = 1, 2, \ldots, n-3\\
&\{q_i, p_n\} = -2\,\mathrm{e}^{q_{n-1}-q_n} + 2\,\mathrm{e}^{q_{n-1}+q_n} \qquad i = 1, 2, \ldots, n-2\\
&\{p_i, p_{i+1}\} = -\mathrm{e}^{q_i-q_{i+1}}(p_i + p_{i+1}) \qquad i = 1, 2, \ldots, n-2\\
&\{p_{n-1}, p_n\} = -(p_{n-1} + p_n)\,\mathrm{e}^{q_{n-1}-q_n} + (p_{n-1} - p_n)\,\mathrm{e}^{q_{n-1}+q_n}
\end{aligned} \tag{39}$$

and all other brackets are zero. Denote this Poisson tensor by J_1 and let J_0 be the standard symplectic bracket. A simple computation leads to the following.

Theorem 4. *The bracket J_1 satisfies:*

1. *J_1 is Poisson.*
2. *J_1 is compatible with J_0.*
3. *The mapping F given by (28) is a Poisson morphism between J_1 and the cubic bracket π_3.*

Thus, in (q, p) coordinates we also have a non-degenerate pair (J_0, J_1) for D_n Bogoyavlensky–Toda and therefore we may define a recursion operator $\mathcal{R} = J_1 J_0^{-1}$. We have then a hierarchy of mutually compatible Poisson tensors defined by $J_i = \mathcal{R}^i J_0$.

The vector field

$$Z_0 = \sum_{i=1}^{n} p_i \frac{\partial}{\partial p_i} + \sum_{i=1}^{n} 2(n-i) \frac{\partial}{\partial q_i} \tag{40}$$

is a conformal symmetry for the Poisson tensors J_0 and J_1 and for the Hamiltonian

$$h_0 = \frac{1}{2} \sum_{1}^{n} p_j^2 + \mathrm{e}^{q_1 - q_2} + \cdots + \mathrm{e}^{q_{n-1} - q_n} + \mathrm{e}^{q_{n-1} + q_n}. \tag{41}$$

We compute

$$\mathcal{L}_{Z_0} J_0 = -J_0 \qquad \mathcal{L}_{Z_0} J_1 = J_1 \qquad \mathcal{L}_{Z_0} h_0 = 2h_0. \tag{42}$$

So Oevel's theorem applies. With $Z_i = \mathcal{R}^i Z_0$, $\chi_0 = \chi_{h_0}$ and $\chi_i = \mathcal{R}^i \chi_0$ one calculates easily that

(a) $[Z_i, \chi_j] = (1 + 2j)\chi_{i+j}$

(b) $[Z_i, Z_j] = 2(j - i)Z_{i+j}$

(c) $\mathcal{L}_{Z_i}(J_j) = (2j - 2i - 1)J_{i+j}$.

Note that (a) gives the exponents (except one) for a Lie group of type D_n.

The action of the first master symmetry on P_n is the same as in Flaschka coordinates:

$$Z_1(P_n) = h_0 P_n. \tag{43}$$

Finally, we calculate that

$$[Z_0, \chi_{P_n}] = (n-1)\chi_{P_n} \tag{44}$$

producing the last exponent.

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